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NOTE

REDUCTIONS OF THE GRAPH RECONSTRUCTION CONJECTURE

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In this note we shall show that the Graph Reconstruction Conjecture (also called the Kelly–Ulam conjecture [1, p. 11]) is equivalent to a conjecture about the algebraic properties of certain directed trees and their homomorphic images. We shall also show that the Graph Reconstruction Conjecture is equivalent to recognizing the (abstract) group of a graph from the tree (generalized "deck") of the graph.

1. The tree of a graph

A *p-permutation tree* is a directed rooted tree with all arcs directed from the root with end points colored either 0 or 1 where the points of such a tree are the non-repeating sequences of integers chosen from $[1, p]$ of length less than or equal to $p-2$ (including the empty sequence $\emptyset$, the root, of length 0), and a point v is adjacent to vk for $k \in [1, p]-v$.

Suppose T is a p-permutation tree such that the color on each end point depends only on the elements of the sequence and not their order. We define a directed rooted acyclic graph T^* called *the quotient of* T as follows. The points of T^* are the subsets of $[1, p]$ of size less than or equal to $p-2$. A point v is adjacent to $v \cup \{k\}$ for $k \in [1, p]-v$. The end points of T^* are colored either 0 or 1 according to the color of any of the corresponding end points of T.

When we consider homomorphisms and isomorphisms between graphs with colored points, we always require colors to be preserved by the maps. If T is a p-permutation tree with quotient T^* there is a natural homomorphism $*$ from T onto T^* which takes a sequence to the set of its elements.

If G is a graph on the p points $1, \ldots, p$ we associate to G the p-permutation tree T_G defined as follows: if $v = v_1, \ldots, v_{p-2}$ is an end point of T_G then v is colored 1 if and only if $G - \{v_1, \ldots, v_{p-2}\}$ is isomorphic to K_2, i.e., has one line. Obviously T_G always has a quotient. If I is a graphical invariant, then I is said to be *recognizable* if two graphs with isomorphic trees have the same value of I.

We shall prove the following theorems.

Theorem 1. *The following statements are equivalent.*

(1) *The Graph Reconstruction Conjecture is true.*

(2) *For any isomorphic p-permutation trees T_1, T_2 with quotients T_1^*, T_2^*, respectively, there are isomorphisms i, i^* such that the following diagram commutes*

$$\begin{array}{ccc} T_1 & \xrightarrow{i} & T_2 \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *} \\ T_1^* & \xrightarrow{i^*} & T_2^* \end{array}$$

Theorem 2. *The following statements are equivalent.*

(1) *The Graph Reconstruction Conjecture is true.*

(2) *The (abstract) group of a graph is recognizable.*

Theorem 3. *The following statements are equivalent.*

(1) *The Graph Reconstruction Conjecture is true.*

(2) *A graph is self-complementary if and only if it and its complement have isomorphic trees.*

Lemma 1. *The following statements are equivalent.*

(1) *The Graph Reconstruction Conjecture is true.*

(2) *Two graphs are isomorphic if and only if their trees are.*

Proof. First observe that any isomorphism of two graphs naturally extends to an isomorphism of their trees upon identifying one element sequences with their elements. Next, let G be any graph and $v = v_1, \ldots, v_k$ a point of T_G; set $G(v) = G - \{v_1, \ldots, v_k\}$. If $vu_1, \ldots, vu_m$ are all the points of T_G adjacent from v, then $G(vu_1), \ldots, G(vu_m)$ is a deck of $G(v)$.

Now suppose (1) and that i is an isomorphism of T_{G_1} onto T_{G_2}; for points v of T_{G_1} we prove by induction on $p-2-\text{length}(v)$ that $G_1(v)$ is isomorphic to $G_2(i(v))$.

Basis: $p-2-\text{length}(v)=0$. We have $K_2 \cong G_1(v) \Leftrightarrow v$ is colored $1 \Leftrightarrow i(v)$ is colored $1 \Leftrightarrow G_2(i(v)) \cong K_2$ and $\bar{K}_2 \cong G_1(v) \Leftrightarrow v$ is colored $0 \Leftrightarrow i(v)$ is colored $0 \Leftrightarrow G_2(i(v)) \cong K_2$.

Induction step: $p-2-\text{length}(v)>0$. Let $vu_1, \ldots, vu_m$ be all the points of T_{G_1} adjacent from v, then $i(vu_1), \ldots, i(vu_m)$ are all the points of T_{G_2} adjacent from $i(v)$. Since $\text{length}(vu_j) = \text{length}(v)+1$, for $1 \leq j \leq m$, by the induction hypothesis each $G_1(vu_j)$ is isomorphic to $G_2(i(vu_j))$. Thus by the second observation above and (1) $G_1(v)$ is isomorphic to $G_2(v)$.

Thus $G_1 = G_1(\emptyset) \cong G_2(i(\emptyset)) = G_2(\emptyset) = G_2$ and (1) implies (2).

Finally suppose (2). For any p-permutation tree T with point v let $T(v)$ be the subtree of T rooted at v. Let G_1 and G_2 be given with point set $[1, p]$ and suppose

π is a permutation of $[1, p]$ such that for $j \in [1, p]$ $G_1 - \{j\}$ is isomorphic to $G_2 - \{\pi(j)\}$. It follows that

$$T_{G_1}(j) \cong T_{G_1 - \{j\}} \cong T_{G_2 - \{\pi(j)\}} \cong T_{G_2}(\pi(j)),$$

so $T_{G_1} \cong T_{G_2}$ hence G_1 is isomorphic to G_2. Thus (2) implies (1).

Lemma 2. *Two graphs are isomorphic if and only if the quotients of their trees are.*

Proof. First observe that any isomorphism of two graphs naturally extends to an isomorphism of the quotients of their trees upon identifying singleton sets with their elements.

Suppose we are given G_1 and G_2 and i^* an isomorphism of $T^*_{G_1}$ onto $T^*_{G_2}$. Again we assume G_1 and G_2 have point set $[1, p]$. Define a permutation π of $[1, p]$ by setting $\pi(j) = k$ if $i^*(\{j\}) = \{k\}$. We shall show for each point v of $T^*_{G_1}$ that $i^*(v) = \pi(v)$, where $\pi(v) = \{\pi(x) : x \in v\}$, by induction on $|v|$.

Basis: $|v| \leq 1$. The basis case holds by definition.

Induction step: $|v| > 1$. Let $v = \{v_1, \ldots, v_m\}$; the points of $T^*_{G_1}$ adjacent to v are just $v - \{v_1\}, \ldots, v - \{v_m\}$ so by the induction hypothesis $i^*(v)$ is adjacent from $\pi(v) - \{\pi(v_1)\}, \ldots, \pi(v) - \{\pi(v_m)\}$. Since $m \geq 2$

$$i^*(v) = \bigcup_{1 \leq j \leq m} (\pi(v) - \{\pi(v_j)\}) = \pi(v).$$

Finally π is an isomorphism from G_1 onto G_2 since $j \operatorname{adj}_{G_1} k \Leftrightarrow [1, p] - \{j, k\}$ is colored 1 in $T^*_{G_1} \Leftrightarrow [1, p] - \{\pi(j), \pi(k)\}$ is colored 1 in $T^*_{G_2} \Leftrightarrow \pi(j) \operatorname{adj}_{G_2} \pi(k)$.

Lemma 3. *Each permutation tree with a quotient is the tree of some graph.*

Proof. Given a p-permutation tree T with a quotient, to define G such that $T = T_G$ define $j \operatorname{adj}_G k \Leftrightarrow [1, p] - \{j, k\}$ is colored 1 in T^*.

We shall now prove Theorem 1.

Proof of Theorem 1. Suppose (1) and let T_1 and T_2 be isomorphic permutation trees with quotients. By Lemma 3 there are G_1 and G_2 such that $T_1 = T_{G_1}$ and $T_2 = T_{G_2}$. By (1) and Lemma 1 G_1 and G_2 are isomorphic. Let $\bar{i}$ be an isomorphism from G_1 onto G_2. Let i be the natural extension of $\bar{i}$ which results from identifying one element sequences with their elements and let i^* be the natural extension of $\bar{i}$ which results from identifying singleton sets with their elements. We have

$$\begin{aligned} i^* * (v_1, \ldots, v_k) &= i^*(v_1, \ldots, v_k) = \{\bar{i}(v_1), \ldots, \bar{i}(v_k)\} \\ &= *(\bar{i}(v_1), \ldots, \bar{i}(v_k)) = *\, i(v_1, \ldots, v_k), \end{aligned}$$

so the appropriate diagram commutes.

Suppose (2). For any G_1 and G_2, if T_{G_1} is isomorphic to T_{G_2} then $T^*_{G_1}$ is isomorphic to $T^*_{G_2}$ so by Lemma 2 G_1 is isomorphic to G_2. Thus (1) follows by Lemma 1.

It is worth observing here that (2) cannot be strengthened to make $*$ into a functor for the appropriate categories. For example, any automorphism of T_{K_4} which involves transposing 1 and 2, and 23 and 12 does not commute with any automorphism of $T^*_{K_4}$.

2. Recognizable invariants

We now consider recognizable invariants. Many invariants are easily proved recognizable, e.g., number of points, number of lines, degree sequence, number of components, connectivity, cycle rank, arboricity, and point independence number. A complete invariant is recognizable if and only if the Graph Reconstruction Conjecture is true. The second theorem says that this is also true for some invariants which are not complete.

In order to prove Theorems 2 and 3 it is useful to reformulate the existence of an isomorphism from the tree of G_1 onto the tree of G_2 as the existence of a "winning strategy" for a game between two players I and II. The game is played as follows. Players I and II alternately choose points from G_1 and G_2 respectively for less than or equal to p-2 moves (player I can stop whenever he wants). For player II to win he must choose points from G_2 in such a way that all possible plays of the game determine an isomorphism from T_{G_1} onto T_{G_2}.

Lemma 4. *If the Graph Reconstruction Conjecture is false, then the (abstract) group of a graph is not recognizable.*

Proof. Suppose G_1 and G_2 have isomorphic trees, the same group, and are not isomorphic. Set $G = G_1 \cup G_1$ and $H = G_1 \cup G_2$. We shall show that G and H have different groups but the same tree.

Since G_1, G_2 can be assumed to be connected, by Theorems 14.5 and 14.6, p. 166 of [1]

$$\Gamma(G) \cong S_2[G_1] \not\cong \Gamma(G_1) + \Gamma(G_1) \cong \Gamma(H).$$

We now show that there is an isomorphism from T_G onto T_H by showing that player II has a winning strategy for the game determined by G and H. Since G_1 and G_2 have isomorphic trees, player II has a winning strategy for the game determined by G_1 and G_2. Obviously, he also has a winning strategy for the game determined by G_1 and G_1. He can win the game determined by G and H by choosing points in the summand corresponding to where I chooses, and choosing according to his winning strategy for the corresponding game.

Theorem 2 follows easily from Lemmas 1 and 4.

3. Self-complementary graphs

We now proceed to prove Theorem 3.

Lemma 5. *If the Graph Reconstruction Conjecture is false then there is a non-self-complementary graph with a tree isomorphic to the tree of its complement.*

Proof. If the Graph Reconstruction Conjecutre is false, since connectivity is recognizable, there are non-isomorphic connected G_1 and G_2 with isomorphic trees. Define a graph $\mathfrak{A}(G_1, G_2)$ as follows. $\mathfrak{A}(G_1, G_2)$ is obtained from $G_1 \cup \bar{G}_2 \cup \bar{G}_2 \cup G_1$ by adding all possible lines between the ith and $i+1$st summand for $1 \leqslant i \leqslant 3$. In [2] we observe that for arbitrary connected G and H with the same number of points $\overline{\mathfrak{A}(G, H)} \cong \mathfrak{A}(H, G)$, and G is isomorphic to H if and only if $\mathfrak{A}(G, H)$ is self-complementary. It remains to show that $\mathfrak{A}(G_1, G_2)$ and $\mathfrak{A}(G_2, G_1)$ have isomorphic trees.

We show that player II has a winning strategy for the game determined by $\mathfrak{A}(G_1, G_2)$ and $\mathfrak{A}(G_2, G_1)$. Since G_1 and G_2 have isomorphic trees player II has winning strategies for the games determined by G_1 and G_2, and $\bar{G}_2$ and $\bar{G}_1$. He can win the game determined by $\mathfrak{A}(G_1, G_2)$ and $\mathfrak{A}(G_2, G_1)$ by choosing points in the summand corresponding to where I chooses, and choosing according to his winning strategy for the corresponding game.

Theorem 3 follows easily from Lemmas 1 and 5.

References

[1] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).
[2] R. Statman and F. Harary, The graph isomorphism problem is equivalent to the legitimate deck problem for regular graphs, to appear.